論　文

On Some Bilateral Exchange Method And Its Dynamics

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* The programs in this paper that are written in Mathematica are available from the author.

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論文要約

Walras的市場における一般均衡を発見する問題は1950年代にArrow-Hurwicz（1958）等によって集中的に研究がなされてきた。その中の最も重要な結果は市場超過需要にgross substitutesという強い条件を付加すればWalras的価格調整プロセスは安定的性質を保証されるということである。しかし、このプロセスが、この強い条件なしに果たして安定であり得るか否かについては依然不明であった。例えば、ArrowとHurwiczは各個人の効用関数の凸性の性質がプロセスの安定性を保証し得るかもしれないという推論を提起していた。

Scarf（1960）は上記の推論に対する否定的解答を二つの反例を示すことによって与えた。その後、理論家はtatonnementから数学的に一般均衡を解く不動点近似問題へと研究の方向を変更させていった。しかし、その結果、Walrasの抽象的市場とその背後に存在する現実の分権的市場との間の関連性を究明するという経済学にとって重要な問題が置き去りにされてしまったかのようにも思われる。このギャップを埋めるための一つの研究方向は、Smith（1965）によって提唱されPlott等によって研究が相当進展しているダブル・オークション分権的市場における実験経済学的手法であろう。事実、Andersonその他（2004）の研究は、分権的市場における取引価格の推移とWalras的価格調整プロセスとの関連性を主張することによて市場均衡価格の現実的意味を明らかにしつつある。

最近Gintis（2007）はAndersonその他と同一の交換モデル（Scarfモデル）を採用して、又コンピューター・シミュレーションの方法により、相対売買取引市場に於ける価格動学を研究し極めて強い命題を主張している。即ち、初期に設定された私的価格分布の平均値は取引経過に伴って徐々に一般均衡価格に収束する。本稿において、この彼の主張は必ずしも正しくないことを独自のプログラムと理論的考察をも加えて例証する。

1. Introduction

The problem of discovering a general equilibrium in Walras central market was intensively in the 1950s studied by Arrow and Hurwicz (1958), Arrow and Block and Hurwicz (1959) and others: The most important result is that when we put a fairly strong condition as "gross substitutes" on the market excess demands, a Walrasian price adjustment process (called tatonnement) is globally stable. Whether or not a tatonnement can be stable without this assumption was left unclear. Also in Arrow and Hurwicz one conjecture had been proposed; a tatonnement process may be possible to be stable without this condition if we well utilize the property of convexity of preferences.

Soon Scarf (1960) gave a negative answer to the above conjecture by showing two counter examples; one is concerned with Leontief type utility function with no substitutability, and another is concerned with C.E.S. type utility function. Later this demonstration seems to be reconfirmed by a series of the study of market excess demand functions which was initiated by Sonnenschein (1972) and further developed by Debreu (1974) and others. Since then, theorists have changed a direction of study about the discovery of general equilibrium from Walrasian tatonnement to computation of solving a fixed point (See Scarf (1972)).

However it seems that there still remains an important study of investigating a relation between an abstract Walras central market and possible decentralized markets which will be behind abstract central market. The double auction market proposed by Smith (1965) and still now continuing to be developed by Plott and many others will be a candidate to decentralized markets. By a series of researches of this market the relation between general equilibrium and decentralized markets like double auction market is now becoming clear. In particular, Anderson, Plott, Shimomura and Granat (2004) suggest us that the path of the average of transaction prices in double auction market looks like following a Walrasian price adjustment process: In their experiments they adopt the model Scarf proposed as an example of instability of tatonnement, and when a model predicts a limit cycle the data in experiment also depicts a kind of limit cycle, and when a model predicts a convergence of equilibrium the data shows similar convergence.

Recently Gintis (2007) has studied the price dynamics in a bilateral decentralized exchange economy based on an agent based simulation and private prices of agents, in which Scarf model is also adopted. In his paper Gintis asserts that a sequence of distribution of private prices and a sequence of the average of private prices will converge to the general equilibrium, while both a tatonnement and a double auction trading process show instability. If his assertion turns out to be true irrespectively of choice of preferences and initial holdings, his method seems to take a position in the study of finding a general equilibrium price.

In this paper we try to follow a method of Gintis and to do several simulation experiments by our own computer program, and to examine whether or not his assertion can be true as a general tendency.

2. Bilateral Exchange Method With Two Goods And Its Dynamics

2.1. Recently Gintis(2007) has attempted to study a class of decentralized market in which a sequence of the averages of transaction prices would tend to converge to the general equilibrium price: His method is mainly due to a distribution of private prices given to all agents at starting time of transaction, and is to show that according as transaction proceeds the distribution function may change and the average of changing distribution function will converge to the equilibrium price.

In his paper Gintis actually discusses two models; (i) Scarf type model of exchange (this type model also has been examined in an experimantal double auction study by Anderson *et al.* (2004)), and (ii) Walrasian production economy. However, in order to clarify whether or not this Gintis' method can be true as a general tendency irrespective of the choice of utility functions and initial endowments, in this paper we want to concentrate on the problem about only exchange model which seems to be fundamental to Gintis' method. In this section we study two goods exchange economy with Gintis' bilateral method, which will show us an intrinsic nature of his method, though only the three goods exchange model was handled in the Gintis' paper.

Let us consider an exchange economy with two types of individuals and two goods which is charactrized by

$$\{\, u^j(x_{1j}, x_{2j}),\ \omega^j = (\omega_{1j}, \omega_{2j}) \,\}\ ,\ j = 1,2,$$

where utility function $u^j(\cdot)$ is assumed to satisfy the conventional assumptions such as convexity and ω^j denotes the initial endowment vector of the j th type.

Let us consider a situation in which there exist n agents of each type, so that totally 2n. For convenience, we number all agents from 1 to 2n, for example, agents of type 1 from 1 to n, agents of type 2 from n+1 to 2n.

Next we proceed to the problem of how agents trade each other, and to describe a Gintis' way of exchange in the starting period, say, period 1.

[trade procedure in period 1]

(i) We take the second good as *numeraire*, so that the price of second good is unity. Let us give all agents private prices by the following way; take supremum and infimum price, say, $\bar{p}$ and $\underline{p}$, and consider an interval $[\underline{p}, \bar{p}]$ and take a set S of lattice points in the above interval. Suppose a private price vector of each agent is randomly chosen from S, say, $\{p(1), p(2), \ldots, p(2n)\}$, which may be considered to be fairly uniformly distributed in S if n is fairly large.

(ii) All agents can be sometimes "initiator" or "responder". Suppose agent i is an initiator and agent j is a responder. In this case agent i only can offer a trade term, that is, an exchange of some amount of good k for amount of good r. On the other hand, agent j only determines whether to accept or reject this trade term offered by agent i.

Next we have to think about how two agents come across, in a market, as initiator or responder. For this matching problem, we want to take a simple way such that an initiator is, in turn from the agents set $\{1,2,\ldots,2n\}$, chosen, and a responder is randomly chosen from the set in which this initiator is excluded. In our experiment we will show later, all agents are assumed to be initiator one time in the simulation in this section and r times in the next section. (In Gintis(2007), r=1 is assumed.)

Next let us consider the problem of agents' behavior patterns.

[behavior pattern of initiator]

agent i determines net demand or net supply by maximizing the utility function subject to the budget constraint with his own private price. Let the stock of good 1 and good 2 of agent i be ω_1 and ω_2. Then the net demand or net supply of good 1 are defined as

$$d_{1i} = E_{1i}(p(i), \omega_1, \omega_2) \quad if\, E_{1i}(\cdot) \; > 0$$

$$s_{1i} = -E_{1i}(p(i), \omega_1, \omega_2) \quad if\, E_{1i}(\cdot) \; < 0$$

where E_{1i} is the agent i's excess demand function of good 1. For example, when $E_{1i}(\cdot) \; > 0$, agent i tries to give an offer $(d_{1i}, s_{2i} = -p(i)d_{1i})$ to a responder which will be chosen as in the above.

[behavior pattern of responder]

The behavior pattern of responder is very simple: Suppose agent i (initiator) give a specific trade term (d_{1i}, s_{2i}) to agent j (responder). If this exchange brings about the non-decrease of the value evaluated by agent j's private price, i.e.,

$$s_{2i} - p(j)d_{1i} \geq 0,$$

he will be willing to accept this offer, and reject otherwise. When agent j has enough stock to supply d_{1i}, this exchange is carried out with no amount change of goods at all, and when not enough stock, an actual trade amounts are assumed to permit to scale down until agent j's stock of good 1.

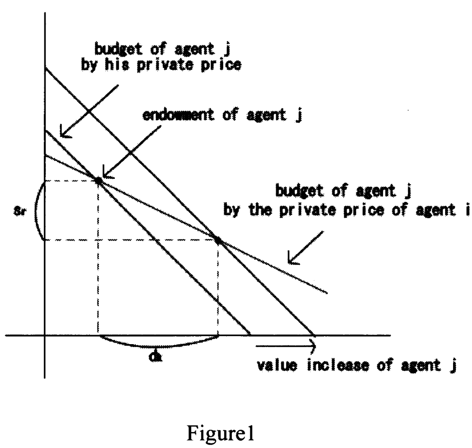

Figure1

Next we proceed to the problem of trading in subsequent periods.

Assumption 1 (Renewal of initial holdings): After trades in a period, the final stock of goods is assumed to be consumed by agents. The same initial holdings as in period 1 are again given agents before starting trade in next period.

It is assumed that the private prices $\{p(1), p(2), \ldots, p(2n)\}$ given in the starting period continue to be kept until t times of periods are finished. And we call t times of periods one generation. Of course, the final stock of agents may change from period to period.

After one generation is over, all agents can calculate the total of utility value over period, and compare those to each other. For example, take agent i and agent j of type 1, and suppose that a sequence of their final stock in period is

$$\{x(i,1), x(i,2), \ldots, x(i,t)\} \text{ and } \{x(j,1), x(j,2), \ldots, x(j,t)\}.$$

When estimating the above by their common utility function and summing up, we have

$$(*) \qquad \text{total utility of agent } i = \sum_{k=1}^{t} u_1(x(i,k))$$
$$(**) \qquad \text{total utility of agent } j = \sum_{k=1}^{t} u_1(x(j,k)).$$

Imitation rule: If (*) > (**), then agent j tries to imitate private price $p(i)$ and vice varsa.

We choose m sets of pair (i,j) of agents randomly from group of each type, and apply the above imitation rule to m sets of pair of each type. This way will make the private prices of 2m agents change.

In addition to the imitation of private prices, we finally put the mutate assumption on change of prices.

Assumption 2 (Mutation): r percentage of all agents mutate, and make their private prices increase or decrease change by a certain percentage.

The Gintis' assertion is as follows.

Gintis' assertion: According as generation proceeds, a possible distribution of private prices that can be obtained at each generation will gradually change from the uniform distribution at the starting period, and the average of distributions at each generation will converge to the general equilibrium price.

2.2. Our concern here is to clarify whether or not this assertion will be true. We want to show two results of our computer simulation that are performed in the programs by Mathematica; (i) a case with Leontief type utility function, which may be convenient when compared to Scarf type utility function with three goods (see next section and Gintis(2007)), and (ii) a case with Cobb-Douglas type utility function that satisfies the conventional assumptions such as strict convexity and smoothness.

Let us first consider a case of Leontief type utility function. The utility functions and the initial endowments of type 1 and 2 are:

$$\begin{aligned} u^1(x) &= \min[\alpha x_1, x_2], \quad \omega^1 = (\omega_1, 0) \\ u^2(x) &= \min[\beta x_1, x_2], \quad \omega^2 = (0, \omega_2) \,. \end{aligned}$$

We have the following market excess demand function of good 1:

$$E_1(p) = p\omega_1/(p+\alpha) + \omega_2/(p+\beta) - \omega_1 \,.$$

A simple calculation tells us: (i) The interior equilibrium price is given by $p^* = \alpha(\beta - \omega)/(\omega - \alpha)$ with $\omega = \omega_2/\omega_1$, (ii) when the relation of parameters is $\alpha < \beta$, $\beta > \omega > \alpha$ or $\alpha > \omega > \beta$, the boundary equilibria exist as $p^{**} = 0$ and $p^{**} = \infty$, and furthermore a tatonnement process $\dot{p} = E_1(p)$ never converges to the interior equilibrium p^*, rather converges to the boundary equilibria, and (iii) when $\alpha > \beta$, $\beta > \omega > \alpha$ or $\alpha > \omega > \beta$, the boundary equilibria do not exist and a tatonnement process converges to the unique interior equilibrium.

We demonstrate the typical cases with (i) $\alpha = 5$, $\beta = 15$, $\omega = 10$, and $p^* = 5$, and (ii) $\alpha = 5$, $\beta = 14$, $\omega = 10$, and $p^* = 4$ in both of which the number of each type is set $n = 100$. The computer simulation of case (i) neglects mutation process while that of case (ii) takes mutation process into consideration.

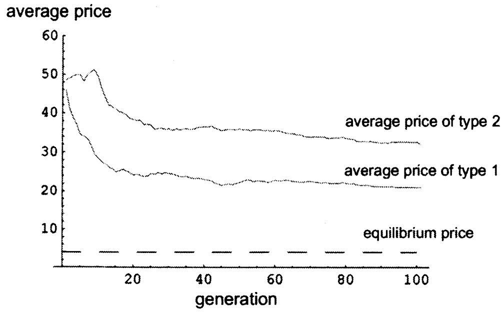

Figure 2: average price path in case (i)

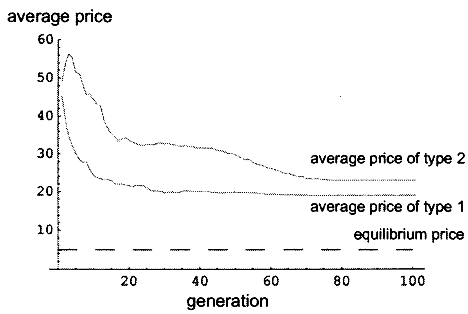

Figure 3: average price path in case (ii)

From those demonstrations we see that a bilateral exchange process proposed by Gintis does not necessarily guarantee the convergence to the equilibrium price, contrary to Gintis' assertion. However, the utility functions in those demonstrations are very extreme (that is, no substitutability among goods at all), and we are afraid that this extremity of utility functions might be the main source of non-convergence of process. Therefore we want to study the case of Cobb-Douglas utility function which satisfies all the conventional conditions.

We suppose the following Cobb-Douglas utility functions of both type and as to initial endowments we follow the same notation in the previous discussion:

$$u^1(x) = x_1^{\alpha} x_2^{1-\alpha}$$

$$u^2(x) = x_1^{\beta} x_2^{1-\beta} .$$

Put $\alpha = 1/3$, $\beta = 1/5$, $\omega = 12$, so that $p^* = 4$. The following picture by our computer simulation demonstrates the average price path in this case with the number of each type being $n = 100$.

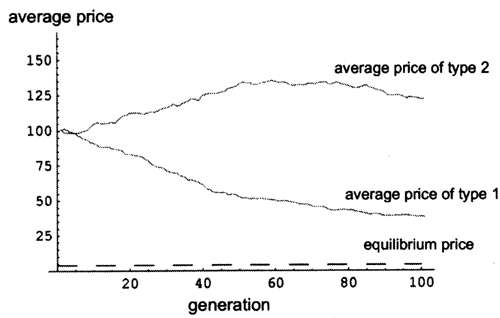

Figure 4: average price path in Cobb-Douglas case

This demonstration again seems not to guarantee the convergence to the general equilibrium price; the smoothness and substitutability of utility function will fail to give us the property of convergence. A question why non-convergence in our cases happens despite of Gintis' demonstration of convergence will remain to be answered.

2.3. Let us proceed to ask the above question. In order to simplify our discussion we define the following density function of private prices: Let the private price of agent i of type 1 be denoted by $p(i)$ with the property that $p(i) < p(i+1)$ and $p(i+1) - p(i) = p(i) - p(i-1)$. Then the uniform density function is

(1) $f(p) = 1/n$ *for* $p = p(i) = i$ or $p = p(n+i) = i$, $i = 1, 2, \ldots, n$

$\quad 0$ otherwise .

The density function of agents of type 2 can be defined as the same form.

Next we consider a class of possible density functions that can be produced based on trading processes according to the evolution of generation. Let the number of imitation pair be m. Then the set of this pair is

$$s(\hat{i},\hat{j}) = \{(i_1,j_1),(i_2,j_2),\ldots,(i_m,j_m)\}$$

where $\hat{i} = \{i_1,i_2,\ldots,i_m\}, \hat{j} = \{j_1,j_2,\ldots,j_m\}$.

A density function at generation 1 for $s(\hat{i},\hat{j})$ can be given by

$$(2) \qquad f^1_{s(\hat{i},\hat{j})}(p) = \begin{cases} 2/n & \text{for } p = p(i_r),\ r = 1,2,\ldots,m \\ 1/n & \text{for } p \neq p(i_r) \neq p(j_r) \\ 0 & \text{otherwise}. \end{cases}$$

Suppose the total utility value of agent i_k be larger than or equal to that of agent j_k. For simplicity we set $m = 1$ in the following discussion. In the next generation (generation 1) the number of density functions is possible to be $n(n-1)$. These density functions can be produced by actual trading process.

Let us consider a simple case with $n = 10$. A density function at generation 1 corresponding to $s(2,1)$ is given by the following picture.

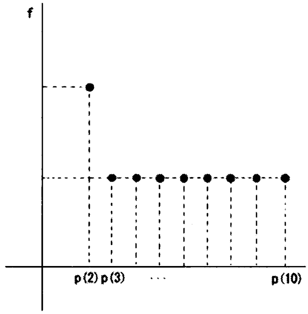

Figure 5 density function $f^1_{s(2,1)}(p)$ at generation 1

This picture is one of the possible class of density functions the number of which is ninety. And the number of possible density functions at generation 2 which are derived from each density function at generation 1 is 72, so that the total patterns are 6480. We can also calculate a class of density functions at subsequent generations.

Below we try to show that any density function which will be possible to produce at any generation has the probability to realize throughout actual trading processes. First let us consider a possible class of density function at generation 1 and take a special density function from this class;

$$f^{1}_{s(2,1)}(p) = \begin{cases} 2/n & \textit{for } p = p(2) \\ 1/n & \textit{for } p \neq p(1) \neq p(2) \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Next we demonstrate a certain case of actual trading process which is supportable for the above density function. At the begining of this period, the stock of initial holdings is assumed to reset, so that if a pair of initiator and responder belongs to the same type, no trade between them happens because they have only the same good (there does not exist an exchange good). When $n = 5$ and $m = 1$, we consider a special matching case as follows:

initiator	1	2	3	4	5	6	7	8	9	10
responder	2	7	4	1	3	8	6	9	8	9

In this case, only one pair (2,7) can trade irrespective of the choice of utility function, but other pair (7,6) can trade when agent 7 wants to demand the second good and can not trade when he wants to demand the first good. Therefore, if this matching case occurs through ten periods (clearly this has a certain probability), them the total utility sum of agent 2 becomes larger than that of any other agent of type 1. This result does not depend on the form of utility function. This tells us that the above density function will be certainly supportable for actual trading process. We are also able to show that even in more general cases any density function will have at least one trading matching pattern.

Proposition 1: Any density function at generation t that will be derived from one density function at generation $t-1$ will be supportable for actual trading pattern in the Gintis type decentralized bilateral exchange market.

From this proposition we are able to see: Suppose the i th agent be given an equilibrium price p^* as his private price. At some generation t, it may be possible that agent i imitates other agent's private price; a density function in this case is given by $f^t_{s(r,i)}$.

Next we want to ask the probability for agent i who has the equilibrium price as his private price not to imitate other private prices. When the number of agents of each type is n, the number of matching patterns is $(n-1)^n$. Because this number is extremely vast if n is quite large, it is very difficult to handle them, so that we will explain the above by using rather simple case with $n = 3$ and $m = 1$. Also assume that the utility funcitons and initial endowments

$$u^1(x) = \min[5x_1, x_2], \quad \omega = (10, 0)$$
$$u^2(x) = \min[15x_1, x_2], \quad \omega = (0, 100)\ .$$

Futhermore we assume the following initial private prices:

$$(p(1), p(2), p(3), p(4), p(5), p(6)) = (4, 5, 6, 4, 5, 6)$$

where $p(2) = p(5) = 5$ is shown to be the equilibrium price.

Let us show the result of our computer simulation. In this setting, the probabilities that agent 2 does not imitate others at generation 1,2 and 3 are respectively 0.447, 0.346 and 0.327. And the probabilities that both agent 2 and 5 do not imitate others at generation1,2 and 3 are respectively 0.14, 0.13 and 0.12.

This result enables us to conclude the following.

Proposition 2: The price dynamics of the Gintis type bilateral exchange market does not necessarily converge to the equilibrium price, rather may converge to other prices except the equilibrium price.

3. Bilateral Exchange Method With Three Goods And Its Dynamics

3.1. In Section 2 we have studied that in the case of two goods the bilateral exchange method proposed by Gintis does not necessarily guarantee the convergence of the average trading prices to the equilibrium price. This will be quite different from the original assertion of Gintis, which has been done by using the famous Scarf exchange model with three goods. In this section we want to study whether or not his assertion based on three goods model can be true as a general tendency.

Let us consider a pure trade model (used in our agent based simulation study) in which there are three commodities and three types of agents. Each type is characterized by utility function and initial holdings, i.e.,

$$\{\ u_j(x_{1j},x_{2j},x_{3j}),\ \omega_j = (\omega_{1j},\omega_{2j},\omega_{3j})\ \}\ ,\ j = 1,2,3$$

where the first subscript means commodity and the second subscript individual. Throughout our study in this chapter we will use C.E.S. utility function and its limit form, i.e.,

$$(4) \qquad u_j(x) = [a_{1j}^{1+\rho}x_{1j}^{-\rho} + a_{2j}^{1+\rho}x_{2j}^{-\rho} + a_{3j}^{1+\rho}x_{3j}^{-\rho}]^{-\frac{1}{\rho}}$$

or in a different form,

$$(4)' \qquad u_j(x) = -[a_{1j}^{1+\rho}x_{1j}^{-\rho} + a_{2j}^{1+\rho}x_{2j}^{-\rho} + a_{3j}^{1+\rho}x_{3j}^{-\rho}]\ .$$

(4) or (4)′ yields the following demand function:

$$(5) \qquad f_{ij}(p,\ m_j) = \frac{a_{ij}\,p_i^{-c}\,m_j}{\sum_{k=1}^{3} a_{kj}\,p_k^{1-c}}\quad ,\ i = 1,2,3;\ j = 1,2,3$$

where $p = (p_1,p_2,p_3)$ denotes a price vector, $m_j = \sum p_j\ \omega_{ij}$ denotes the nominal income of individual j , and $c = 1/(1 + \rho)$ is the elasticity of substitution. It is well known that when $c = 0$, (4) becomes Leontief type utility function with no substitutability among goods, and when $c = 1$, (4) becomes Cobb-Douglas type utility function.

From (5) we have the market excess demand functions as follows:

$$(6) \qquad E_i(p) = \sum_{j=1}^{3} f_{ij}(p, m_j) - \sum_{j=1}^{3} \omega_{ij} \ , \quad i = 1,2,3.$$

Since the model we will use in first experiment is basically the one Scarf (1960) proposed as instability example of tatonnement, we want to review it briefly.

Let us put a special value on the parameter matrix in (4) ;

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

and also put a special value on the endowment matrix ;

$$\omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The above values imply that all individuals consume only two goods, either of which they holds initially. Scarf (1960) showed that when $c = 0$, that is, utility function is of the perfectly complementary, a tatonnement process $\dot{p}_i = E_i(p)$ depicts a limit cycle, never converges to the equilibrium price vector (also see, Hirota(1981)).

Recently Anderson *et al.* (2004) used this Scarf model in their double auction experimental study. Gintis (2007) also used it in a bilateral exchange experimental economy with private price setting.

In Scarf's original model, $p^* = (1,1,1)$ is the equilibrium price vector up to multiplication, but both Anderson *et al.* and Gintis, from the view point of application, change the measure of goods so as the equilibrium price of goods to be different:

$$A = \begin{bmatrix} 0 & \gamma & \gamma \\ \alpha & 0 & \alpha \\ \beta & \beta & 0 \end{bmatrix} \quad \textit{and} \quad \omega = \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \alpha \\ \beta & 0 & 0 \end{bmatrix} .$$

When the measure change is the above, Scarf model may be described as follows;

$$\text{(7)} \qquad \begin{aligned} &\text{type 1:} \quad u_1(x) = \min[\tfrac{x_2}{\alpha}, \tfrac{x_3}{\beta}] \ , \ \omega_1 = (0,\ 0,\ \beta) \\ &\text{type 2:} \quad u_2(x) = \min[\tfrac{x_1}{\gamma}, \tfrac{x_3}{\beta}] \ , \ \omega_2 = (\gamma,\ 0,\ 0) \\ &\text{type 3:} \quad u_3(x) = \min[\tfrac{x_1}{\gamma}, \tfrac{x_2}{\alpha}] \ , \ \omega_3 = (0,\ \alpha,\ 0) \ . \end{aligned}$$

In this case the equilibrium price vector becomes $p^* = (\beta/\gamma, \beta/\alpha, 1)$ with third good taken as *numeraire*, it is also easy to see that a tatonnement process depicts limit cycle. In their experiments, $\gamma = 20,\ \alpha = 40,\ \beta = 800,$ so that $p^* = (40,\ 20,\ 1)$.

In our analysis we will handle a case with $c > 0$, that is, smooth utility function, so that now we want to clarify the relationship between the parameter value in A and the above equilibrium p^* (however, the endowment matrix is kept as the same value).

Define

$$\text{(8)} \qquad a_{1j} = b_{1j}\gamma^{1-c} \ , \ a_{2j} = b_{2j}\alpha^{1-c} \ , \ a_{3j} = b_{3j}\beta^{1-c} \ ,$$

and define the coefficient matrix of b_{ij}

$$\text{(9)} \qquad B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix} .$$

By a simple calculation we are able to see that when B is doubly stochastic, $p^* = (\beta/\gamma,\ \beta/\alpha,\ 1)$ is always the equilibrium price vector irrespective of the values of b_{ij}'s and the value of c.

3.2. Let us explain the method Gintis (2007) proposed in the simulation study of dynamics of bilateral exchange economy based on agents' private prices. For private price soon we will clarify it. In his paper he has studied not only exchange economy but production economy. However, for exchange economy, he has studied only Scarf economy. We want to handle not only Scarf economy, but also C.E.S. economy, and etc. Therefore we will mention more general method, following Gintis' method.

By mainly taking Scarf model (7) into account, we want to describe our exchange procedure in a bilateral exchange economy. The method we are going to say may be extended to more general model such as the C.E.S. economy.

Let us consider a situation in which there exist n agents of each type, so that totally 3n. For convenience, we number all agents from 1 to 3n, for example, agents of type 1 from 1 to n, agents of type 2 from n+1 to 2n, agents of type 3 from 2n+1 to 3n.

Next we proceed to the problem of how agents trade each other, and to describe a way of exchange in the starting period, say, period 1.

[trade procedure in period 1]

(I) Let us give all agents private prices by the following way; take supremum and infimum price, say, $\bar{p}$ and $\underline{p}$, and consider a rectangular

$$\{ (p_1,p_2) |\ \underline{p} \leq p_i \leq \bar{p},\ i = 1,2 \}$$

and take a set S of lattice points in the above rectangular. Suppose a private price vector of each agent is randomly chosen from S, say, $\{ p(1), p(2),\dots, p(3n) \}$, which may be fairly uniformly distributed in S.

(II) All agents can be sometimes "initiator" or "responder". Suppose agent i is initiator and agent j responder. In this case agent i only can offer a trade term, that is, an exchange of some amount of good k for amount of good r . On the other hand, agent j only determines whether to accept or reject.

Next we have to think about how two agents match in a market. For this matching problem, we want to take a simple way, all agents become initiator in turn, and others are randomly chosen responder. In our experiment we will show later, all agents are assumed to be initiator r times. (In Gintis (2007), $r = 1$ is assumed.)

(III) In this stage we will say about behavior patterns as initiator and as responder.

(behavior pattern of responder)

The behavior pattern of responder is very simple: Suppose agent i responder and initiator agent j offer an exchange s_r for d_k. If the value of agent i by this exchange increases based on his given private prices, i.e.,

$$p_k(i)d_k - p_r(i)s_r \geq 0 ,$$

this agent i tries to accept this offer, and to reject otherwise. Thoughout trading process we put a rule that permits agents to scale down this offer $(d_k,\ s_r)$.

(behavior pattern of initiator)

Below we want to name the good for only exchange T good, the good for consumption W good, and the good held initially H good.

1. (i) When this agent j has some amount of T good, say, ω_T, first he tries to exchange ω_T for W good based on his private price $p(j)$. Because of the same value exchange, the demand of W good determined by

$$\text{the demand for W good: } d_W = \left(\frac{p_T(j)}{p_W(j)}\right)\omega_T .$$

If responder accepts this offer (which may be scaled down by a rule), this trade is executed and stock of goods of both agents is changed. On the other hand, if responder rejects this offer, initiator think of next offer.

(ii) Initiator offers an exchange of ω_T for H good to the same responder. If responder accepts this offer, this trade is executed. Otherwise, the trade of both agents is never done.

2. (i) When an initiator has no T good, this agent i determines net demands for W good and H good by maximizing utility function subject to budget constraint with private prices $p(i)$. Let the stock of H good and W good of agent i be ω_H and ω_W. Then the net demands are defined as

$$d_k = f_{ki}(p(i), \omega_H, \omega_W) - \omega_k \quad , k = W, H ,$$

where $f_{ki}(\cdot)$ is the demand function of good k. If $d_W > 0$, agent i tries to exchange

$s_H = -d_H$ for d_W and if $d_H > 0$, agent i tries to exchange $s_W = -d_W$ for d_H. Because of $p_W d_W + p_H d_H = 0$, if $d_W > 0$, agent i tries to exchange $s_H = -\left(\frac{p_W(i)}{p_H(i)}\right)d_W$ for d_W and if $d_H > 0$, agent i tries to exchange $s_W = -\left(\frac{p_H(i)}{p_W(i)}\right)d_H$ for d_H.

If responder accepts this offer (which may be scaled down by a rule), this trade is executed between two agents and stock of goods of both agents is changed. On the other hand, if responder rejects this offer, initiator think of next offer.

(ii) An initiator, agent i offers an exchange of half amount of s_W or s_H for T good to the same responder. If rejected, the trade between both agents is never done.

Remark: Gintis (2007) says that whenever holding H good agent i always plans to exchange H good for W good by utility maximization (See p.1286 in Gintis (2007)). However, as we have shown above, this supposition may be possible to be contradictory.

Here it may be useful to give an example of a trade process in Scarf economy (7).

Consider agent i of type 1 first appears in this trading market as an initiator. So he has only the stock of H good, β (in this case $H = 3$), and according to the behavior pattern 2, he tries to exchange s_3 for d_2. s_3 and d_2 are determined by maximizing utility $\min[x_2/\alpha, x_3/\beta]$ subject to budget constraint $p_2(i)d_2 + p_3(i)d_3 = 0$;

$$d_2 = \frac{\alpha\beta}{\alpha p_2(i)+\beta} \quad \textit{and} \quad s_3 = \left(\frac{p_2(i)}{p_3(i)}\right)d_2 = \frac{\alpha\beta p_2(i)}{\alpha p_2(i)+\beta} .$$

Let agent j be a responder with the stock of good 2. According to the simple behavior pattern of responder, this agent has to examine an arbitrage condition which tells whether to increase the value after trade.

The arbitrage condition is:

$$\frac{p_2(j)}{p_3(j)} \geq \frac{p_2(i)}{p_3(i)} . \tag{10}$$

Remark: In our experiment the equality may be important because otherwise no agents can trade at all in a case with general equilibrium price p^* being given all agents.

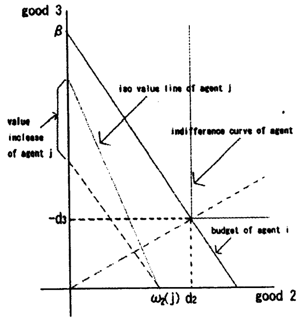

Figure 6

Let us suppose that arbitrage condition is satisfied but the stock of agent j , $\omega_2(j)$, is less than the demand of agent i . But by the scaling down rule, a trade of $\lambda(d_2, s_3)$ with $0 < \lambda < 1$ is executed between agents i and j .

Figure 6 well demonstrates this fact.

3.3. Next we proceed to the problem of trading in subsequent periods.

Assumption 3 (renewal of initial holdings): After trades in a period, the final stock of goods is assumed to be consumed by agents. The same initial holdings as in period 1 are again given agents before starting trade in this period.

It is assumed that the private prices $\{ p(1), p(2), \ldots, p(3n) \}$ given in the starting period continue to be kept until t times of periods are finished. And we call t times of periods one generation. Of course, the final stock of agents may change from period to period.

After one generation is over, all agents can calculate the total of utility value over period, and compare those to each other. For example, take agent i and agent j of type 1, and suppose that a sequence of their final stock in period is

$$\{ x(i,1),\ x(i,2),\ \ldots,\ x(i,t) \}\ \textit{and}\ \{ x(j,1),\ \ldots,\ x(j,t) \}$$

When estimating the above by their common utility function and summing up, we have

(*) total utility of agent $i = \sum_{k=1}^{t} u_1(x(i,k))$

(**) total utility of agent $j = \sum_{k=1}^{t} u_1(x(j,k))$.

Imitation rule: If (*) > (**), then agent j tries to imitate private price $p(i)$ and vice varsa.

We choose m sets of pair (i, j) of agents randomly from group of each type, and apply the above imitation rule to m sets of pair of each type. This way will make the private prices of 3m agents change.

In addition to the imitation of private prices, we finally put the mutate assumption on change of prices.

Assumption 4 (Mutation): r percentage of all agents mutate, and make their private prices increase or decrease change by a certain percentage.

As in the above, we can imagine that, according as generation proceeds, the distribution of private prices would gradually change from the uniformity in the starting period. Our concern is to pursue its dynamics.

3.4. The computer program written based on the trading method in section 3 produces a sequence of price distribution in each generation as follows:

$$\{ p(1,t),\ p(2,t),\ \ldots,\ p(3n,t) \}\ ,\ t = 1,2,\ldots$$

where t means the time of generation. In this section we want to study whether or not the above sequence may converge to the equilibrium price p^*. Gintis (2007) has asserted the following:

Gintis' assertion: A sequence of average of prices in each generation,

$$\hat{p}_i(t) = \frac{\sum_{k=1}^{3n} p_i(k,t)}{3n}\ ,\ i = 1,2\ ; t = 1,2,\ldots\ ,$$

may converge to the equilibrium price p_i^*.

When we presume Scarf economy with $\gamma = 20$, $\alpha = 40$, $\beta = 800$, our program has produced the result below. In this experiment we set the parameters as follows;

(parameter case 1): number of each type $n = 800$, the round time in period $r = 2$, the number of periods in one generation = 10, the times of generation $t = 1000$, percentage of imitation = 10%, and percentage of mutation = 3% with private prices being increased or decreased by 10%.

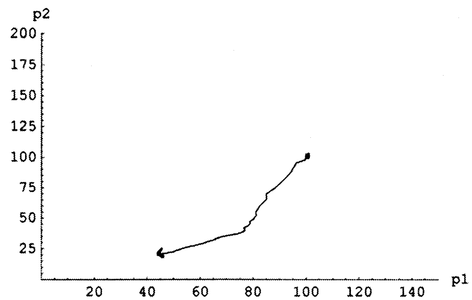

figure 7 (parameter case 1)

As long as we see this picture, it seems that a sequence of average prices may converge to the equilibrium price, $p_1^* = 40$, $p_2^* = 20$. However this picture does not tell us any information about the variance of distribution when the distribution is changing toward the future. This question leads us to examine what will happen to the movement of sequences of prices in each type. Let us consider:

$$\hat{p}_i(1,t) = \frac{\sum_{k=1}^{n} p_i(k,t)}{n}, \ \hat{p}_i(2,t) = \frac{\sum_{k=n+1}^{2n} p_i(k,t)}{n}, \ \hat{p}_i(3,t) = \frac{\sum_{k=2n+1}^{3n} p_i(k,t)}{n}.$$

Those paths of each average are shown in the next picture:

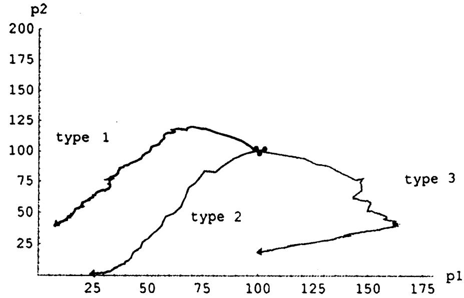

figure 8

What does this picture tell us? Three paths for each type looks converging to completely different points which are far from the equilibrium point. A question is why whole average path shows us the convergence to equilibrium. This seems to be an important point we have to resolve.

[Question]: Why does the total average path converge to equilibrium regardless of different convergence of each type of path?

The data of the above experiments might be accidentally obtained. Therefore we conducted several experiments with the same parameter case 1. However, the data obtained showed quite similar results. The question is still left to be unclear.

We may summarize these experiments' result:

Result 1: When the number of agents is so large, the total path of prices may be possible to converge to the equilibrium, but each type of path is seen not to converge to it.

In order to clarify the phenomenon of these data, we turn to a slightly different experiment from the above: We presume number of agents fairly smaller than case 1, that is, n = 30, and furthermore we are afraid that because of small agents the uniform distribution of initial private prices may not be guaranteed, so that we give 10 % of agents the equilibrium price as their private prices.

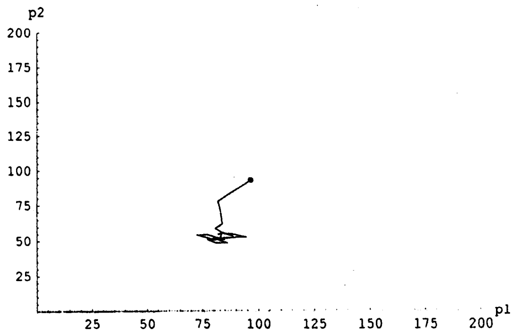

figure 9

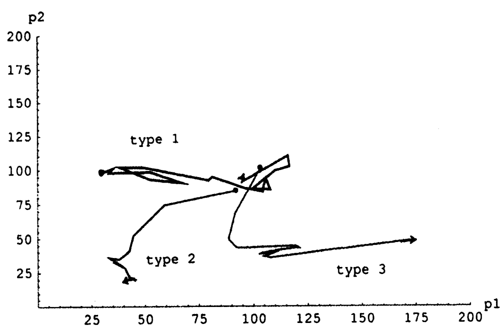

figure 10

Figure 9 shows the price dynamics of total average prices and figure 10 shows the price dynamics of average prices of each type. This experiments may be interesting because total path never shows convergence, rather looks like cycle. Each type path goes toward completely different directions. In particular the path of type 3 is diverging rather than converging.

Result 2: When the number of agents is not so large rather small, both total path and each type path are seen not to converge to the equilibrium.

By the observation of the data in the above we have seen that our trade method of bilateral exchange economy does not necessarily give the stable function of price discovery process. One of the reason for that might be due to the assumption of mutation, which always causes some fluctuation of prices even in equilibrium. Next we try to do an experiment without this assumption and we have a result below. In this experiment, the number of each type of agents is set $n = 200$, and the remaining parameters are the same as in first experiment.

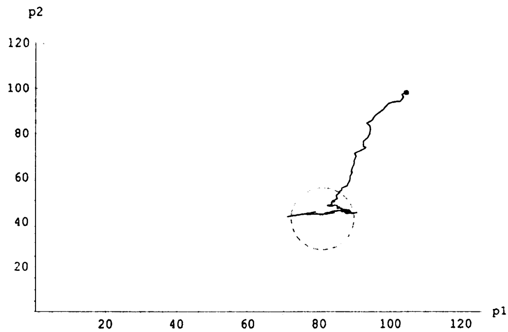

figure 11

As long as we observe the data in figure 11, the total average path of all agents is not seen to move to the equilibrium price. When we observe the data in a circle region in the above picture, the path looks like depicting a limit cycle. On the other hand, the paths of each type agents looks like diverging from initial average prices points, and never converging to the equilibrium point. This phenomenon may be common to in the previous experiments that we have already demonstrated.

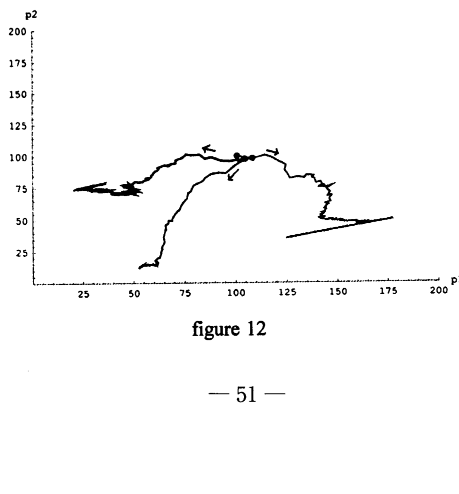

figure 12

We have observed an interesting phenomenon in which the price distribution in the last generation completely converges to the respective points of types which are different each other. Since there is no mutation in this experiment, this is a natural consequence.

3.5. So far we have studied that sequences of the average prices and distribution of prices do not necessarily predict some convergence, contrary to the assertion of Gintis. Therefore we have to pursue why this will happen. The reasons we can imagine now are: (i) This phenomenon may be mainly due to a special form of utility funcion (that is, Leontief type of utility function with no substitutability), or (ii) due to various assumption of trading method we have followed a way of Gintis, and etc. In this section we try to handle our bilateral exchange economy with C.E.S. utility function instead of Scarf type of utility function.

In several experiments in this case we set various parameters as follows: The number of each type agents $n = 300$, the round of periods in one generation is 10, the number of generation is 500, imitation percentage is 10%, and mutation percentage is 1.3%, and furthermore, the elasticity of substitution that is common to all agents is $c = 0.5$.

We have observed that several experiments under the above setting of parameters bring about a certain of phenomena that is fairly common to all attempts. Here we want to demonstrate typical case which will be very interesting.

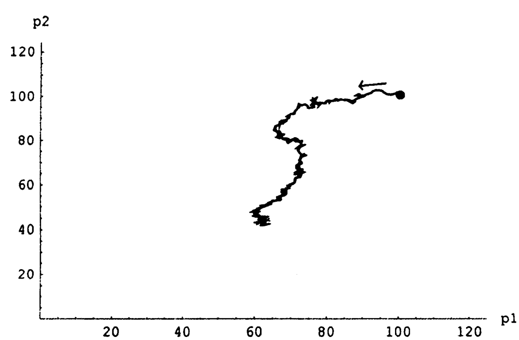

figure 13

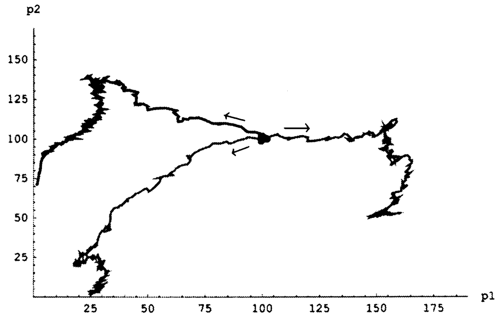

figure 14

Figure 13 is concerned with the path of average price of total agents while Figure 14 is concerned with the paths of average prices of each type of agents. From the observation of data we are able to see that the path of average prices of total agents looks going to the certain point $(p_1, p_2) = (62, 42)$ which is fairly far from an equilibrium price $p^* = (40, 20)$. Also the paths of average prices of each type agents are going to completely different points, the path of type 1 to a point $p = (1, 71)$, the path of type 2 to a point $p = (25, 2)$, and the path of type 3 to a point $p = (159, 52)$.

Subsequently we tried to do similar experiments with the same parameters except a value of elasticity of substitution. In this experiments we set a value of elasticity of substitution $c = 1$, that is, Cobb-Douglas utility function case. However, we could not obtain a particular result different from the one of the previous experiments with $c = 0.5$. We are now able to state:

Result 3: Even when we assume C.E.S. utility function with substitutability among goods, the paths of average prices of all agents and each type of agents may be very possible not to predict a certain convergence of an equilibrium.

Next we examine what happens to a Walrasian price adjustment in this C.E.S. economy with $c = 0.5$. The basic model in the last experiment is as follows:

The parameter value in C.E.S. utility function of each type is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & \gamma^{1-c} & \gamma^{1-c} \\ \alpha^{1-c} & 0 & \alpha^{1-c} \\ \beta^{1-c} & \beta^{1-c} & 0 \end{bmatrix} .$$

In the above, each value of the coefficient matrix B in (6) is all set $b_{ij} = 1$ so as the equilibrium price to be $p^* = (\beta/\gamma,\ \beta/\alpha,\ 1)$. And the endowment matrix of all types is also set

$$\omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} = \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & \alpha \\ \beta & 0 & 0 \end{bmatrix} .$$

When the rate of change of prices is assumed to be proportionate to the market excess demands, it can be represented by the following differential equation

$$\text{(11)} \qquad \begin{aligned} \dot{p}_1\ (t) &= E_1(p) = f_{12}(p, m_2) + f_{13}(p, m_3) - \gamma \\ \dot{p}_2\ (t) &= E_2(p) = f_{21}(p, m_1) + f_{23}(p, m_3) - \alpha \end{aligned}$$

where $m_1 = \beta$, $m_2 = p_1\gamma$, $m_3 = p_2\alpha$. We want to consider whether or not (11) can be stable. To do so let take Jacobian of (11) evaluated around the equilibrium $p^* = (\beta/\gamma,\ \beta/\alpha,\ 1)$;

$$J(p^*) = \begin{bmatrix} -0.13 & 0.38 \\ -0.13 & -0.50 \end{bmatrix} .$$

The eigen values of $J(p^*)$ are $(-0.31 + 0.11i,\ -0.31 - 0.11i)$, which implies (11) is locally stable. Since a price adjustment process (11) converge to equilibrium, (11) can find equilibrium by using public price in central market. However as is seen in this section, a complex trading method we assume seems to fail to find an equilibrium.

4. Final Remarks

A bilateral exchange method with agent based simulation proposed by Gintis may be stimulating for the further researches, in particular, (i) the relationship between tatonnement and non-tatonnement process (See Hahn and Negishi (1962)) and various decentralized markets such as double auction markets, and (ii) the relationship between Gintis type bilateral exchange method and other type bilateral exchange (for example, see Foley (1994)) and etc.

However, unfortunately, as is seen in this paper, this Gintis' method does not necessarily guarantee to discover equilibrium price contrary to Gintis' assertion, what are the main sources to bring about this non-convergence phenomenon? One reason may be clarified in Section 2.

At the starting period the private prices of all agents may be considered to be fairly uniformly distributed, and according to the proceed of periods, some of private price will disappear gradually by the imitation process we suppose, and if some private price that is given initially disappeared through trading process, such a price would never appear again. Therefore if some price very close to the equilibrium price disappears, such a price can not be restored to possible agent price since then. This may be due to the fact that Gintis method does not take a new creation process of prices into consideration. In order to improve this defection and obtain the stable function of bilateral exchange market, we have to adopt a fairly new method which will be different from Gintis.

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