Article

Which Does City Size Distribution Depend on, the Rank-Size Rule or the Lognormal Distribution Model?

Takashi INOUE*

Abstract

Models that attempt to describe patterns of city size distribution are classified largely into two types, i.e., the rank-size rule and the lognormal distribution model. Although the logical consistency of the both models had not been sufficiently discussed until 1980s, the author proved in 1998 that the both models are not mathematically compatible with each other and proposed the following hypothesis: urban population by cluster depends on the rank-size rule; urban population by municipality depends on the lognormal distribution model. The purpose of this paper is to examine the hypothesis.

Data of two kinds of urban population were calculated using small area population statistics of Japan census 2005 and GIS, and then the above two models were applied to those data. As a result of this application, we have found that urban population by cluster fits almost completely into the rank-size rule and that urban population by municipality has a relatively high possibility of fitting into the lognormal distribution model.

1. Introduction

As is well known, city size distribution shows almost stable equilibrium in many countries or regions[1]. Models that attempt to describe patterns of such distribution are classified largely into two types: one is the rank-size rule that

---

* Professor, Department of Public and Regional Economics, College of Economics, Aoyama Gakuin University

1) Okabe (1979; 1980), Haag and Weidlich (1984), Sheppard (1985), Tabuchi (1986) and others have strictly discussed some conditions that city size distribution reaches stable equilibrium.

refers to the relationship between rank and size of urban population; the other is the lognormal distribution model that formulates frequency distribution of urban population.

The former model, the rank-size rule means that the size of urban population and its rank depend on Pareto distribution (Suzuki, 1980, pp. 371–372). Pareto distribution introduced by Pareto at the end of 19th century is well-known as a model illustrating income distribution. The size of urban population and its rank show a log-linear relationship if these depend on Pareto distribution. Auerbach (1913) focused first on the log-linear relationship and Zipf (1949) formulated the relationship as a law (Suzuki, 1985, pp. 55–60; Morikawa, 1990, pp. 110–112; Hayashi, 1991, pp. 62–70; Reed, 2002; Soo, 2005). As mentioned later, it has been confirmed that the law holds for many countries/regions although there are differences in goodness-of-fit.

The latter model, the lognormal distribution model giving a lognormal shape to frequency distribution of urban population was proposed first by Gibrat (1931). Thereafter relatively few studies have applied the model to regional population (e.g., Eeckhout, 2004), although many studies have applied it to several phenomena such as size distribution of firms and individual distribution of animals (Crow and Shimizu, 1988).

Few studies had referred to the relationship between the both models until 1980s. Beckmann (1958) considered the models to be the starting point of a discussion on city size distribution. Nevertheless he did not mention the relationship between them. Berry (1961) suggested that the rank-size rule holds for city size distribution if the distribution shows a "truncated" lognormal shape (Suzuki, 1980, pp. 389–390). Shinozaki (1951) and Parr and Suzuki (1973) indicated that the models are approximately consistent with each other under a condition. Before those studies, however, Lotka (1941) insisted that the rank-size rule should be derived from one of Pearson distribution models, which are formulas different from the lognormal distribution model[2)].

As mentioned above, although the rank-size rule and the lognormal distribution model are considered to be basic models on city size distribution, the logical consistency of the both models had not been sufficiently discussed until 1980s. Inoue (1998), however, proved that the both models are not mathematically compatible with each other[3)]. Furthermore the author proposed the following hypothesis as a solution for such incompatibility: urban population by cluster, which means a population agglomeration area demarcated irrespective of municipal boundary, fits into the rank-size rule; urban population by municipality fits into the lognormal distribution model. The purpose of this paper is to examine the above Inoue's hypothesis.

We calculate the above two kinds of urban population using small area population statistics of Japan census 2005 and GIS. GIS technique is largely required for calculation of the population, especially urban population by cluster. As far as the author knows, no previous studies on city size distribution have dealt with population data like urban population by cluster.

This paper reconsiders the study of Inoue (1998) in Chapter 2, examines Inoue's hypothesis in Chapter 3, and describes conclusion in Chapter 4.

2. The Consistency of two Models on City Size Distribution

2–1. Formulas of two Models

This section introduces formulas of the two models and results of studies on them.

2–1–1. The Rank-Size Rule

The rank-size rule formulated by Zipf (1949) is shown in the following equation:

$$p = k\,r^{-a}. \tag{1}$$

Here p indicates population size of a city whose rank is the rth place; a and k are parameters. In general, k denotes population size of a primate city. By taking logarithm of both sides of equation (1) and replacing $\log k$ with b, we get equation (2).

2) The Pearson-type family of distributions means the probability density function $f(x)$ satisfying the following differential equation and is classified into twelve types according to several conditions:

$$\frac{df(x)}{dx} = \frac{(x+a)\,f(x)}{bx^2 + cx + d}$$

Here a, b, c and d are parameters. Lotka (1941) insisted that the rank-size rule is derived from the sixth or eleventh type of the above family.

3) The compatibility/incompatibility between the both models has been actively discussed since 1990s (e.g., Reed and Jorgensen, 2005; Giesen *et al.*, 2010)

$$\log p = -a \log r + b \tag{2}$$

This equation enables us to understand that there is a linear relationship between $\log p$ and $\log r$. Parameter a means a slope of the line and decreases with population dispersion. Simon (1955) indicated that the model is applied to not only city size distribution but also several phenomena treated in ecology, biology and economics. Furthermore, many geographical studies have showed that the model holds for many countries/regions (Isard, 1956; Stewart, 1958; Berry and Garrison, 1958; Berry, 1961; Clark, 1967; Kohsaka, 1978; Suzuki, 1981, 1982; Kitagawa, 1985; Kawabe *et al.*, 1988; Kojima and Hataya, 1995).

2–1–2. The Lognormal Distribution Model

Suppose that logarithm $q(= \log p)$ of population size p satisfies a normal distribution $N(\mu, \sigma^2)$, i.e., q depends on the probability density function (3), the city size distribution becomes a lognormal distribution.

$$g(q) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(q-\mu)^2}{2\sigma^2}\right\} \tag{3}$$

Consequently, the probability density function of population size p is expressed as follows (Crow and Shimizu, 1988, pp. 2–3):

$$\begin{aligned} f(p) &= \frac{1}{\sqrt{2\pi}\sigma p} \exp\left\{-\frac{(\log p-\mu)^2}{2\sigma^2}\right\} \qquad (p > 0), \\ f(p) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad (p \leq 0). \end{aligned} \tag{4}$$

The lognormal distribution is actually defined in the range of $p > 0$, and skewed left unlike a normal distribution (Crow and Shimizu, 1988, pp. 1–12). Berry (1961) did not show the formula of a "truncated" lognormal distribution on which he insisted, however the probability density function of the distribution can be expressed on the analogy of equation (4). For non-negative constant p_0, the function is given by

$$f(p) = \frac{1}{p}\exp\left\{-\frac{(\log p-\mu)^2}{2\sigma^2}\right\}\Bigg/\int_{p_0}^{\infty}\frac{1}{p}\exp\left\{-\frac{(\log p-\mu)^2}{2\sigma^2}\right\}dp \quad (p \geq p_0), \quad (5)$$

$$f(p) = 0 \qquad (p < p_0).$$

Constant p_0 indicates a point where the function is truncated.

Gibrat (1931) insisted that the lognormal distribution model was valuable by applying it to population of metropolitan areas in Europa. Furthermore he theoretically considered the reason why metropolitan population was suited to the model (Suzuki, 1985, pp. 11–18). Nevertheless since he insisted, relatively few studies have applied it to regional population except for Suzuki's papers applying it to population density by region (Suzuki, 1960a, 1960b, 1978).

2–2. Examination of the Mathematical Consistency of two Models

To examine the mathematical consistency between the two models, this section derives the probability density function $g(q)$ of logarithm q of population size p from the formula of the rank-size rule. The fact that the function shows frequency distribution of city size enables us to examine the consistency by comparing it with the lognormal distribution model. This section attempts to investigate variation of the function before deriving it.

2–2–1. Investigation of Variation of Probability Density Function

If city size distribution is suitable for the rank-size rule, what type of variation does the probability density function $g(q)$ show? First, to find a solution to the question, we transform equation (2) into a formula using variable q and make r the subject of the formula:

$$r = \exp\left(\frac{b-q}{a}\right). \quad (6)$$

Since variable q is defined in the non-negative range, the range of rank r is given by $[1, \exp(b/a)]$.

Secondly let Δq be a positive and adequately small constant, and let $n(q)$ be the number of cities such that logarithm q is included in an interval, equation (6) leads to the following formula:

$$n(q) = \exp\left(\frac{b-q}{a}\right) - \exp\left(\frac{b-q-\varDelta q}{a}\right). \tag{7}$$

Consequently, for two real numbers q_1 and q_2 on variable q and for an arbitrary positive constant $\varDelta q$, the following formula holds.

$$n(q_2) - n(q_1) = \exp(b/a)\{\exp(-q_2/a) - \exp(-q_1/a)\}\{1 - \exp(-\varDelta q/a)\} < 0 \tag{8}$$

Furthermore, suppose that the probability density function $g(q)$ satisfies $g(q_1) < g(q_2)$ for a pair of real numbers q_1 and q_2 ($q_1 < q_2$), a positive constant $\varDelta q$ satisfying an inequality $n(q_1) < n(q_2)$ exists necessarily. This inequality, however, is contradictory to inequality (8). Therefore for a pair of real numbers q_1 and q_2 ($q_1 < q_2$), the following inequality holds:

$$g(q_1) \geq g(q_2). \tag{9}$$

Inequality (9) means that the probability density function $g(q)$ decreases monotonically with variable q increasing. On the other hand, it is obvious that equation (3), i.e., a normal distribution, does not draw a monotonically decreasing curve. Therefore variable p does not depend on any lognormal distribution (equation (4)). As a result, it is examined that there is no mathematical consistency between the rank-size rule and the lognormal distribution model.

2–2–2. Derivation of Probability Density Function

The number of all target cities of the rank-size rule is given by $\{\exp(b/a) - 1\}$ since the number is equivalent to the length of the range of rank r in equation (6). Accordingly the proportion of frequency $n(q)$ (equation (7)) to $\{\exp(b/a) - 1\}$ expresses probability of extracting cities included in an interval $[q, q+\varDelta q]$ from all ones. Hence if $\varDelta q$ approaches zero unlimitedly, the proportion equals $g(q)\varDelta q$ and equation (7) leads to the following formula:

$$g(q) = \lim_{\Delta q \to 0} \frac{n(q)}{\{\exp(b/a) - 1\}\Delta q}$$

$$= \frac{1}{\exp(b/a) - 1} \lim_{\Delta q \to 0} \frac{1}{\Delta q} \left\{ \exp\left(\frac{b-q}{a}\right) - \exp\left(\frac{b-q-\Delta q}{a}\right) \right\}. \quad (10)$$

Necessarily equation (10) holds for the range of q in equation (6), i.e., $[0, b]$, and $g(q)$ equals zero in the range less than zero or more than b.

As a result, a formula (equation (11)) of the probability density function $g(q)$ is derived from equation (10).

$$g(q) = \frac{\exp\{(b-q)/a\}}{a\{\exp(b/a) - 1\}} \qquad (0 \le q \le b), \quad (11)$$

$$g(q) = 0 \qquad (q < 0 \quad \text{or} \quad b < q).$$

Equation (11) shows that variable q depends on a negative exponential distribution and that q does not satisfy a normal distribution (equation (3)). Therefore, we conclude again that there is no mathematical consistency between the rank-size rule and the lognormal distribution model.

However, if the left side of a normal distribution curve is largely truncated, the distribution obviously approximates that of negative exponential according to the rightward shift of a truncated point. Consequently if logarithm q draws such a "truncated" curve, in other words, if population size p satisfies equation (5) for an adequately large constant p_0, p is likely to depend on the rank-size rule. Parr and Suzuki (1973) examined to what extent distribution of p was suitable for the rank-size rule under the above conditions using numerical simulation. Shinozaki (1951) also made almost the same argument without simulation.

Incidentally, we can obtain the cumulative distribution function of q from equation (11) by integrating it, or directly from equation (7). The number of cities included in an interval $[0, q]$ is shown in $[\exp(b/a) - \exp\{(b-q)/a\}]$ using equation (7), so that the proportion $G(q)$ of $[\exp(b/a) - \exp\{(b-q)/a\}]$ to the number of all cities is given by

$$G(q) = \frac{\exp(b/a) - \exp\{(b-q)a\}}{\exp(b/a) - 1}. \quad (12)$$

The cumulative distribution function is equivalent to the proportion $G(q)$. Here both of $G(0) = 0$ and $G(b) = 1$ hold.

2–3. Derivation of Hypothesis

The preceding section led to the fact that population size p depends on the rank-size rule if satisfying a "truncated" lognormal distribution in the range more than an adequately large constant p_0. We can derive the following new findings from the above fact: in general, city size distribution fits into the lognormal distribution model and furthermore fits into the rank-size rule if limited to large cities. Inoue (1998) developed the findings into the above-mentioned hypothesis, which is written as follows: urban population by cluster is suitable for the rank-size rule; urban population by municipality is suitable for the lognormal distribution model.

The reason for inferring the latter part of the hypothesis is that the lognormal distribution model was reported to be well applied to prefectural population in Japan and state population in U.S. (Suzuki, 1960a, 1960b, 1978). On the other hand, the reason for inferring the former part is described below. If population by municipality, which we can ordinarily obtain, satisfies the lognormal distribution model, it satisfies the rank-size rule only regarding large cities. This is because the number of small-size municipalities is much insufficient to account for the rank-size rule. The number of population clusters, however, increases steeply with reduction of its size since there are many small-size ones in a municipality. As a result, the former part of hypothesis is strongly inferred.

3. Application of Two Models

This chapter examines the Inoue's hypothesis by applying the two models to two kinds of urban population data.

3–1. Data

This section explains how to calculate the data. First, we extract high-density areas from small area population statistics of Japan census 2005. The small area treated in this study means so-called *chocho-aza* that is much smaller than a municipality such as city, town and village, and that is larger

Table 1 Basic Statistics on Common Logarithms of Urban Population

population data		n	mean	variance	max	min	skewness	kurtosis
population by cluster	1,000 persons / km^2	4,862	3.057	0.553	7.486	2.000	1.004	4.228
	2,000 persons / km^2	5,229	3.046	0.514	7.452	2.000	0.988	4.229
	4,000 persons / km^2	6,047	2.995	0.424	7.396	2.000	1.052	4.781
population by municipality	1,000 persons / km^2	1,860	4.133	0.648	5.925	2.013	−0.136	2.315
	2,000 persons / km^2	1,652	4.078	0.748	5.925	2.000	−0.153	2.169
	4,000 persons / km^2	1,380	3.996	0.879	5.923	2.000	−0.137	2.009

than a census enumeration district. We adopt the following three density criteria to extract high-density areas: 1,000, 2,000, and 4,000 persons per square kilometers. The highest density criterion corresponds to that of Density Inhabited District (DID), which is a statistical area introduced in 1960 to demarcate actually urbanized areas.

Secondly, urban population by cluster is calculated. The population is defined as that of a population agglomeration area that consists of adjoining high-density small areas and that is demarcated irrespective of municipal boundary. We calculate the population by adding up that of extracted small areas using GIS.

Thirdly population by municipality is calculated. The population is defined as the total population of high-density small areas located at the same municipality. We calculate the population by adding up that of extracted small areas using GIS in similar to population by cluster.

Thus we can obtain two kinds of urban population from three sets of population data of high-density small areas, and consequently we gain six sets of urban population data. Table 1 shows basic statistics on common logarithm of urban population. The reason why the minimums in Table 1 conform to almost the same value of 2.0 is that we set those of population 100 to remove clusters/ municipalities whose population was extremely small from analysis[4]. Regarding population by cluster, data size n increases with increase of density criterion; whereas regarding population by municipality, n decreases with increase

4) As for population by municipality (1,000 persons / km^2), the reason why the minimum of logarithm is slightly larger than 2.0 is that the minimum population does not equal to 100.

of density criterion. This is because, with increase of the criterion, clusters were fractionated and municipalities including high-density small areas were reduced.

3–2. Application of the Rank-Size Rule

This section makes scatter diagrams between common logarithm of rank and population size before statistical analysis. If urban population satisfies the rank-size rule, the diagrams should draw a straight line. Figures 1, 2 and 3 are scatter diagrams between common logarithm of rank and population by cluster. All of these diagrams show an almost straight line similar with each other except for the difference of horizontal positions of the right ends. This suggests that population by cluster almost entirely satisfies the rank-size rule. Figures 4, 5 and 6 are scatter diagrams between common logarithm of rank and population by municipality. All of these diagrams show an almost straight line in the range where logarithm of population is over 5.0, whereas show a steeply declining curve in the range under 5.0. These curves are similar with each other except for the difference of horizontal positions of the right ends. This suggests that population by municipality does not well satisfies the rank-size rule.

Figure 1 Double Logarithmic Graph of Rank and Population by Cluster (density criterion: 1,000 persons / km^2)

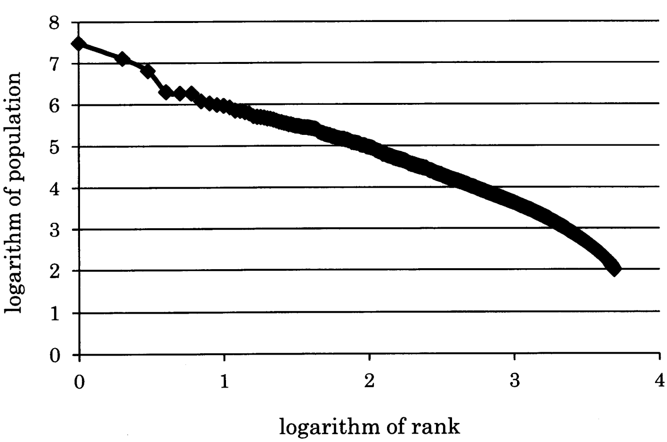


Figure 2　Double Logarithmic Graph of Rank and Population by Cluster
(density criterion: 2,000 persons / km²)

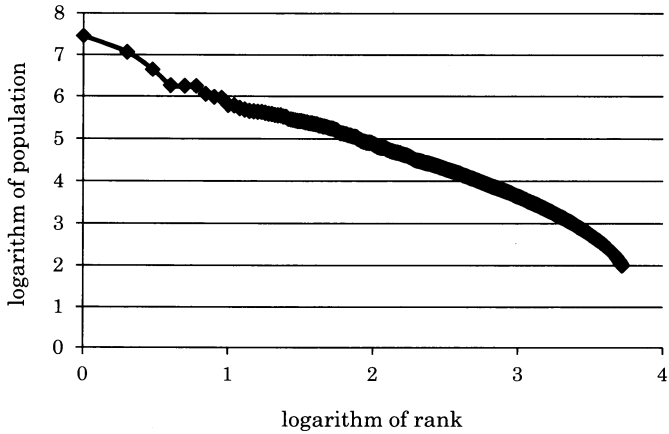


Figure 3　Double Logarithmic Graph of Rank and Population by Cluster
(density criterion: 4,000 persons / km²)

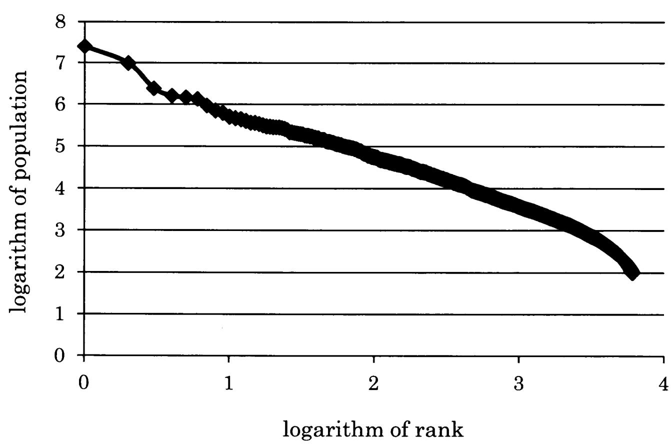


Figure 4 Double Logarithmic Graph of Rank and Population by Municipality (density criterion: 1,000 persons / km^2)

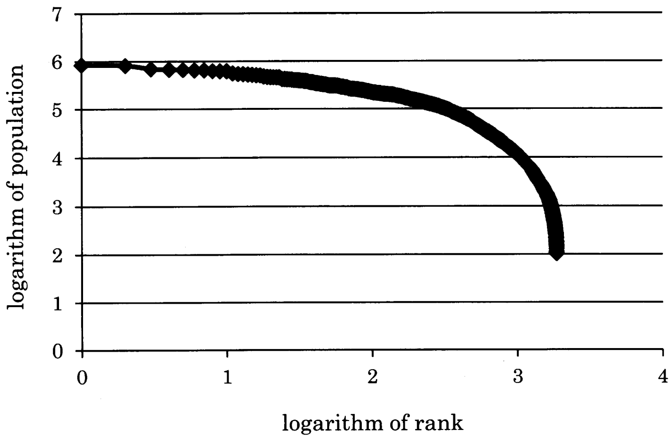


Figure 5 Double Logarithmic Graph of Rank and Population by Municipality (density criterion: 2,000 persons / km^2)

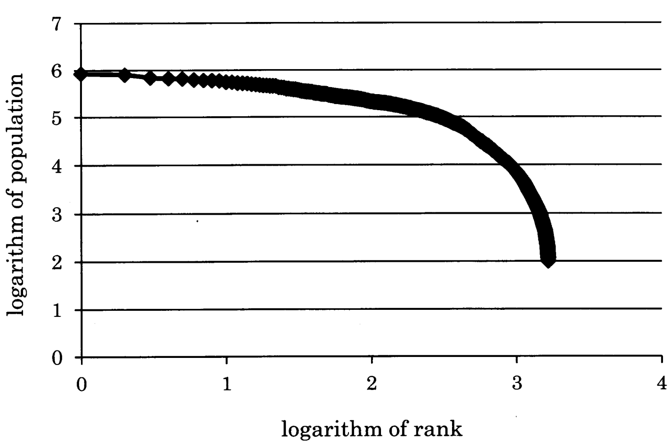


Figure 6 Double Logarithmic Graph of Rank and Population by Municipality (density criterion: 4,000 persons / km²)

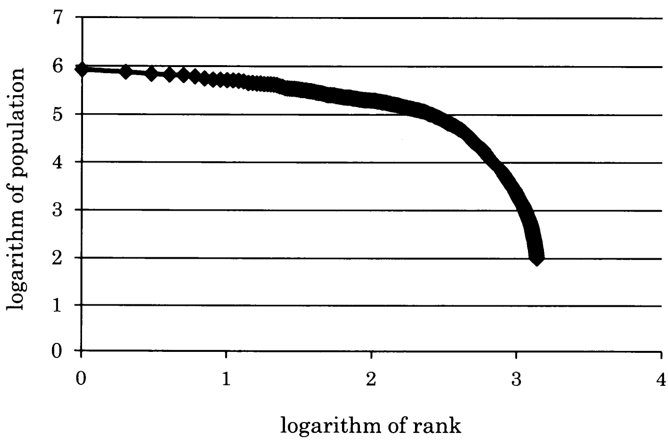


Table 2 Goodness-of-Fit of Regression Formulas

population data		r	adjusted R^2
population by cluster	1,000 persons / km²	−0.9794	0.9592
	2,000 persons / km²	−0.9784	0.9572
	4,000 persons / km²	−0.9794	0.9592
population by municipality	1,000 persons / km²	−0.8765	0.7681
	2,000 persons / km²	−0.8696	0.7561
	4,000 persons / km²	−0.8647	0.7476

We measured goodness-of-fit of the rank-size rule using regression analysis and show results of the analysis in Table 2. Here r indicates a coefficient of correlation. According to Table 2, population by cluster represents very high R-square values; nevertheless population by municipality represents relatively low R-square values. Hence we can conclude that urban population by cluster fits almost completely into the rank-size rule.

3–3. Application of the Lognormal Distribution Model

This section makes histograms of logarithm of population size before statistical analysis. If urban population satisfies the lognormal distribution model, the histograms should draw a symmetric and bell-shaped curve of a normal distribution. Figures 7, 8 and 9 are histograms of logarithm of population by cluster. Since all of these histograms show a distinctly left-skewed distribution, it is clear that population by cluster does not depend on the lognormal distribution model. Figures 10, 11 and 12 are histograms of logarithm of population by municipality. In contrast to the above three figures, skewness is scarcely found in these histograms and week kurtosis is found except for the case of 4,000 persons / km^2.

As regards skewness and kurtosis measured on distribution of urban population, Table 1 gives us more precise values. If the distribution of common logarithm of urban population thoroughly conforms to a normal distribution, the coefficients of skewness and kurtosis become zero and 3.0 respectively. As shown in Table 1, the skewness of population by municipality approaches zero; whereas population by cluster deviates from zero. The kurtosis of popu-

Figure 7 Histogram of Logarithm of Population by Cluster (density criterion: 1,000 persons / km^2)

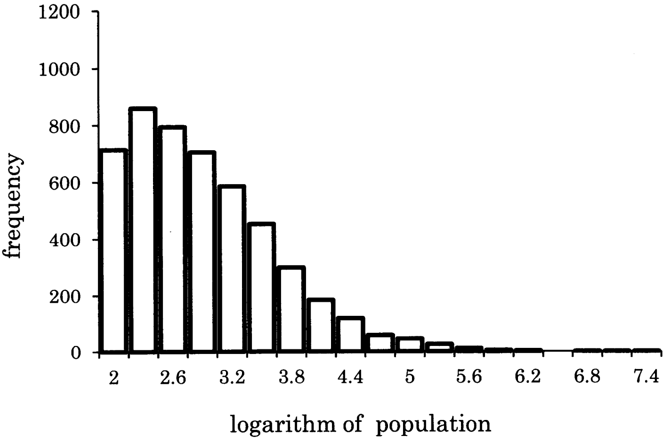


Figure 8 Histogram of Logarithm of Population by Cluster
(density criterion: 2,000 persons / km²)

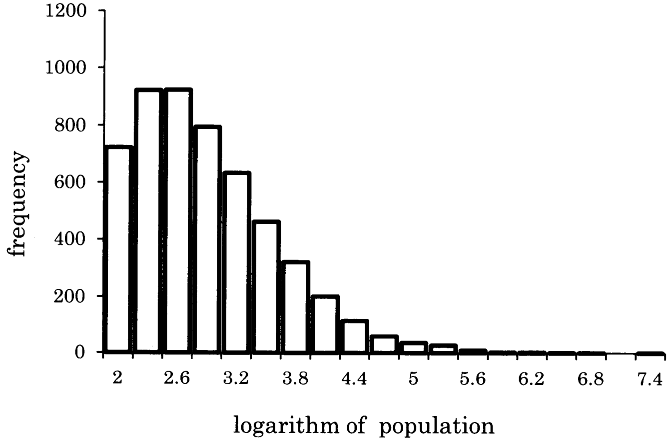


Figure 9 Histogram of Logarithm of Population by Cluster
(density criterion: 4,000 persons / km²)

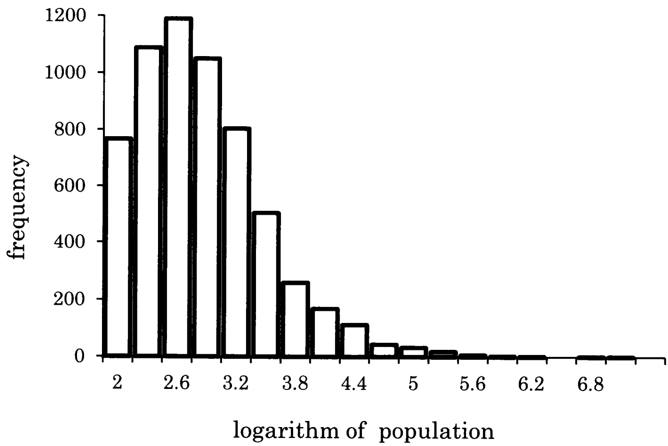


Figure 10 Histogram of Logarithm of Population by Municipality
(density criterion: 1,000 persons / km^2)

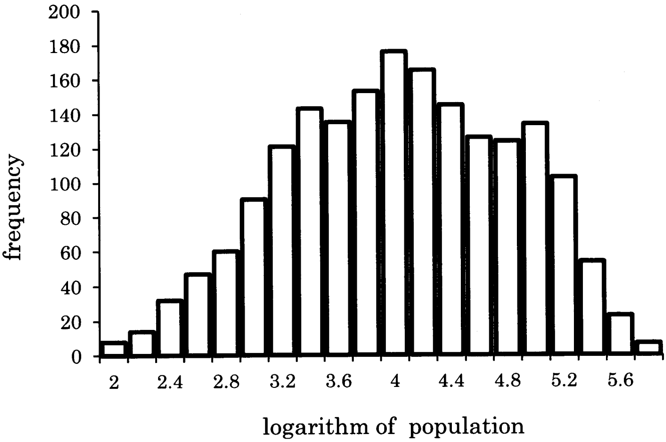


Figure 11 Histogram of Logarithm of Population by Municipality
(density criterion: 2,000 persons / km^2)

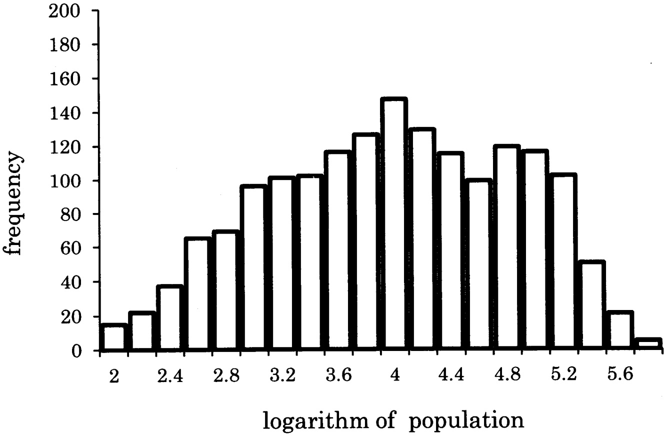
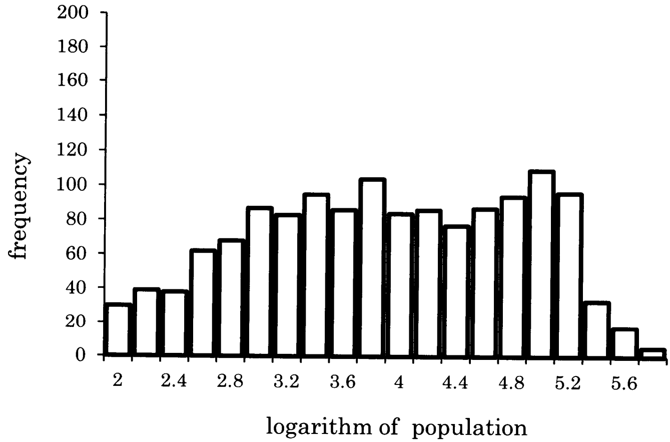


Figure 12 Histogram of Logarithm of Population by Municipality
(density criterion: 4,000 persons / km²)



lation by municipality is, especially in the case of 1,000 persons / km², closer to 3.0 than that of population by cluster.

Furthermore, we set three kinds of normality tests, i.e., skewness test, kurtosis test and Kolmogorov-Smirnov test. The null hypotheses of these three tests are described as follows: the coefficient of skewness equals zero; that of kurtosis equals 3.0; and the target distribution corresponds to a normal distribution, respectively. Table 3 shows results of the former two tests of them. According to two-sided p-values in Table 3, the null hypothesis of skewness was accepted with significant level of 0.01 for three sets of data of population by municipality; that of kurtosis was accepted with significant level of 0.01 only for one set of data of population by municipality. As a result of the Kolmogorov-Smirnov test, the null hypothesis was rejected for all the six sets of data. Those results of tests permit us to conclude that urban population by municipality has a relatively high possibility of fitting into the lognormal distribution model.

4. Conclusion

The purpose of this paper is to examine the hypothesis proposed by the author in 1998. The hypothesis is written as follows: urban population by clus-

Table 3 Results of Normality Test

population data		skewness test			kurtosis test		
		z-statistic	two-sided p-value	one-sided p-value	z-statistic	two-sided p-value	one-sided p-value
population by cluster	1,000 persons / km^2	7.2994	0.0000	0.0000	incalculable		
	2,000 persons / km^2	7.3404	0.0000	0.0000	incalculable		
	4,000 persons / km^2	7.0181	0.0000	0.0000	12.4608	0.0000	0.0000
population by municipality	1,000 persons / km^2	2.2850	0.0223	0.0112	2.2773	0.0228	0.0114
	2,000 persons / km^2	2.3878	0.0170	0.0085	2.8374	0.0045	0.0023
	4,000 persons / km^2	2.0269	0.0427	0.0213	3.0873	0.0020	0.0010

ter is suitable for the rank-size rule; urban population by municipality is suitable for the lognormal distribution model. This paper calculated these two kinds of urban population using small area population statistics of Japan census 2005 and GIS, and then applied the above two models to the urban population.

Regression analysis and three kinds of normality tests were made to measure goodness-of-fit of the rank-size rule and the lognormal distribution model, respectively. The results of these statistical analyses are summarized as follows:

1) In regression analysis, population by cluster represents very high R-square values; population by municipality represents relatively low R-square values.
2) In normality tests, the null hypothesis of skewness was accepted for population by municipality; the null hypothesis of kurtosis was accepted partially for population by municipality; As regards the Kolmogorov-Smirnov test, the null hypothesis was rejected for both of two kinds of urban population.

These results allow us to conclude that population by cluster fits almost completely into the rank-size rule and that population by municipality has a relatively high possibility of fitting into the lognormal distribution model. Therefore this paper has nearly proved the hypothesis proposed by the author.

Acknowledgements

Grateful acknowledgement is made of a grant from the College of Economics of Aoyama Gakuin University in support of the research on which this paper is based.

References

Auerbach, F., 1913. Das Gesetz der Bevölkerungskonzentration. *Petermann's Geographische Mitteilungen*, Vol. 59, pp. 74–76.

Beckmann, M. J., 1958. Hierarchies and the distribution of city size. *Economic Development and Cultural Change*, Vol. 6, pp. 243–248.

Berry, B. J. L., 1961. City size distribution and economic development. *Economic Development and Cultural Change*, Vol. 9, pp. 573–588.

Berry, B. J. L. and Garrison, W. L., 1958. Alternate explanations of urban rank-size relationships. *Annals, Association of American Geographers*, Vol. 48, pp. 83–91.

Clark, C., 1967. *Population Growth and Land Use*. Macmillan, London.

Crow, E. L. and Shimizu, K., 1988. *Lognormal Distribution: Theory and Applications*. Marcel Dekker, New York.

Eeckhout, J., 2004. Gibrat's law for (all) cities. *American Economic Review*, Vol. 94, pp. 1429–1451.

Gibrat, R., 1931. *Les Inégalités Économiques*. Recueil Sirey, Paris.

Giesen, K., Zimmermann, A. and Suedekum, J., 2010. The size distribution across all cities: double Pareto lognormal strikes. *Journal of Urban Economics*, Vol. 68, pp. 129–137.

Haag, G. and Weidlich, W., 1984. A stochastic theory of interregional migration. *Geographical Analysis*, Vol. 16, pp. 331–357.

Hayashi, N., 1991. *Location and Spatial System of Cities*. Taimeido. (in Japanese)

Inoue, T., 1998. The consistency of two laws on city size distribution: Rank-size rule and lognormal distribution model. *Notes on Theoretical Geography*, No. 11, pp. 1–8. (in Japanese)

Isard, W., 1956. *Location and space-economy*. MIT Press, Cambridge, Massachusetts.

Kawabe, H., Kurihara, H., Naito, M., Sato, T. and Sato, K., 1988. *Urban System in Developing Countries*. Institute of Developing Economies. (in Japanese)

Kitagawa, K., 1985. The urban system in India. Yamaguchi, T. ed., *Urban System in the World: An Attempt of New Geography*. Kokon Shoin. (in Japanese)

Kohsaka, H., 1978. Dynamic analysis of city size distribution: A case study of Niigata Prefecture in Japan. *Geographical Review of Japan*, Vol. 51, pp. 223–234. (in Japanese)

Kojima, R. and Hataya, N. eds., 1995. *Urbanization and Poverty Class in Developing Countries*. Institute of Developing Economies. (in Japanese)

Lotka, A. J., 1941. The law of urban concentration. *Science*, Vol. 94, p. 164.

Morikawa, H., 1990. *Urbanization and Urban System*. Taimeido. (in Japanese)

Okabe, A., 1979. Population dynamics of cities in a region: Conditions for a state of simultaneous growth. *Environment and Planning A*, Vol. 11, pp. 609–628.

Okabe, A., 1980. Stable state conditions of population-dependent migration models under a zero natural growth rate. *Journal of Regional Science*, Vol. 20, pp. 353–364.

Parr, J. B. and Suzuki, K., 1973. Settlement populations and the lognormal distribution. *Urban Studies*, Vol. 10, pp. 335–352.

Reed, W., 2002. On the rank-size distribution for human settlements. *Journal of Regional Science*, Vol. 42, pp. 1–17.

Reed, W. and Jorgensen, M., 2005. The double Pareto-lognormal distribution: A new parametric model for size distribution. *Communications in Statistics*, Vol. 34, pp. 1733–1753.

Sheppard, E., 1985. Urban system population dynamics: Incorporating nonlinearities. *Geographical Analysis*, Vol. 17, pp. 47–73.

Shinozaki, K., 1951. Notes on the law of geometrical progression. *Physiology and ecology*, Vol. 6, pp. 127–144. (in Japanese)

Simon, H. A., 1955. On a class of skew distribution functions. *Biometrika*, Vol. 42, pp. 425–440.

Soo, K., 2005. Zipf's law for cities: a cross-country investigation. *Regional Science and Urban Economics*, Vol. 35, pp. 239–263.

Stewart, J. C. T., 1958. The size and spacing of cities. *Geographical Review*, Vol. 48, pp. 222–245.

Suzuki, K., 1960a. The Gibrat distribution of the population densities in Japan. *Medicine and Biology*, Vol. 57, pp. 27–29. (in Japanese)

Suzuki, K., 1960b. The Gibrat distribution of the population densities in the U.S.A.. *Medicine and Biology*, Vol. 57, pp. 189–191. (in Japanese)

Suzuki, K., 1978. Mathematical analysis of population distribution and migration. *Mathematical Sciences*, No. 176, pp. 10–17. (in Japanese)

Suzuki, K., 1980. *Spatial Demography: The Second Volume*. Taimeido. (in Japanese)

Suzuki, K., 1981. On the decomposability of the Zipf's rank-size rule. *Studies in Regional Science*, Vol. 12, pp. 35–52. (in Japanese)

Suzuki, K., 1982. On the homogeneous structure found in the system of the population of cities in Japan. *The Journal of Population Studies*, Vol. 5, pp. 49–56.

Suzuki, K., 1985. *Structural Analysis of Population Distribution*. Taimeido. (in Japanese)

Tabuchi, T., 1986. Existence and stability of city-size distribution in the gravity and logit models. *Environment and Planning A*, Vol. 18, pp. 1375–1389.

Zipf, G. K., 1949. *Human Behavior and the Principle of Least Effort*. Addition Wesley Press, Cambridge, Massachusetts.